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# COMPARISON OF TRUNCATION ERRORS OF OPTIMUM STEADY-STATE AND CRITICALLY DAMPED SECOND-ORDER FILTERS

by

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# COMPARISON OF TRUNCATION ERRORS
# OF OPTIMUM STEADY-STATE AND CRITICALLY
# DAMPED SECOND-ORDER FILTERS

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## ABSTRACT

Second-order, linear, recursive, fixed-weight optimal and critically damped filters are compared on the basis of truncation errors produced by each. The critically damped and optimal filters are derived along with the commonly used parameters which are truncation error and noise ratio. It is shown that when truncation errors are compared on the basis of equal noise ratios for the two filters, the optimal filter yields a significant improvement over the critically damped filter.

# CONTENTS

# ILLUSTRATIONS

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## 1. Critically Damped Second-Order Filters

By definition, a second-order filter is one which will converge on any noise-free second-degree polynomial and yield error-free predictions. The equations involved in a second-order filter containing filter weights of G, H, and K are:

$$\bar{y}_n = \hat{y}_n + G\left(x_n - \hat{y}_n\right) \tag{1}$$

$$\bar{\dot{y}}_n = \hat{\dot{y}}_n + \frac{H}{T}\left(x_n - \hat{y}_n\right) \tag{2}$$

$$\bar{\ddot{y}}_n = \hat{\ddot{y}}_n + \frac{2K}{T^2}\left(x_n - \hat{y}_n\right) \tag{3}$$

$$\hat{y}_{n+1} = \bar{y}_n + \bar{\dot{y}}_n T + \bar{\ddot{y}}_n \frac{T^2}{2} \tag{4}$$

$$\hat{\dot{y}}_{n+1} = \bar{\dot{y}}_n + \bar{\ddot{y}}_n T \tag{5}$$

$$\hat{\ddot{y}}_{n+1} = \bar{\ddot{y}}_n \ . \tag{6}$$

The X's $\iota_{\cdot}$, $\iota_{\cdot}$ as of any of many possible filter inputs, for examp' $\cdot$ radar parameters such as X, Y, Z, range, azimuth, and elevation, while the Y's represent the outputs. In order to distinguish between the smoothing equations and the prediction equations, (^) over a letter indicates predicted values and (-) indicates smoothed values.

The above six equations may be manipulated so that the predicted value at time n + 1 is a function of only the last three measurements (filter inputs), the last three predicted values, and some combinations of the filter weights. This equation is in the form of the general equation for linear feedback filters, Equation (8), and can be written as:

$$\hat{y}_{n+1} = (G + H + K)\,x_n + (-2G - H + K)\,x_{n-1} + G\,x_{n-2}$$
$$- (-3 + G + H + K)\,\hat{y}_n - (3 - 2G - H + K)\,\hat{y}_{n-1}$$
$$- (-1 + G)\,\hat{y}_{n-2} \ . \tag{7}$$

$$\hat{y}_{n+1} = \sum_{i=0}^{N-1} a_i x - \sum_{i=0}^{M-1} b_i \hat{y}_{n-i} \ . \tag{8}$$

1

It is noteworthy that $a_i$ and $b_i$ are given in terms of the weights G, H, and K, and M and N are both three.

The z-transform of Equation (7) may now be taken.

$$Y_n(z) = X_n(z) \, [ \, (G + H + K) + (-2G - H + K) z^{-1} + G z^{-2} ] / [ \, 1$$
$$+ (-3 + G + H + K) z^{-1} + (3 - 2G - H + K) z^{-2} + (-1 + G) z^{-3} ] \, .$$

$$(9)$$

## 2. Optimum Second-Order Filters

From Equation (9) the relations among the weights of a critically damped second-order filter, may be obtained. For a second-order filter to be critically damped, the three roots of the denominator polynomial must be real and equal. The three weighting coefficients (G, H, K) are functions of the roots $(r_1, r_2, r_3)$ of the denominator polynomial.

A term, $\beta$, may be defined as follows:

$$\beta = \frac{1}{r}, \quad r = r_1 = r_2 = r_3 \, ,$$

where r is the real root of the denominator polynomial. The weights are related to r (and $\beta$) by equating the coefficients of like powers of $z^{-1}$ in the cubic $(z^{-1} - r)^3$, to those of the denominator polynomial. The resulting equations are:

$$G = 1.0 - \beta^3$$
$$H = 1.5 (1 - \beta^2) (1 - \beta)$$
$$K = 0.5 (1 - \beta)^3 \, .$$

It can readily be shown that $\beta$ is a good indication of the degree of smoothing. No smoothing will take place if $\beta = 0$, and heavy smoothing will take place as $\beta$ approaches 1. Therefore, many $\beta$'s may be chosen between zero and one, and the critically damped weights solved for. The relationship between the weights may be seen in upper case letters in Figure 1.

Emphasis will now be shifted toward deriving a Kalman steady-state filter from which the weights will be computed for the optimum second-order filter. To begin with, certain parameters must be known in order to initiate the process. One of these parameters characterizes the expected maneuver of the target. It should again be pointed out that the filter operates on only one measured coordinate at a time, for example, range. Therefore, a maneuver covariance matrix, Q, such that
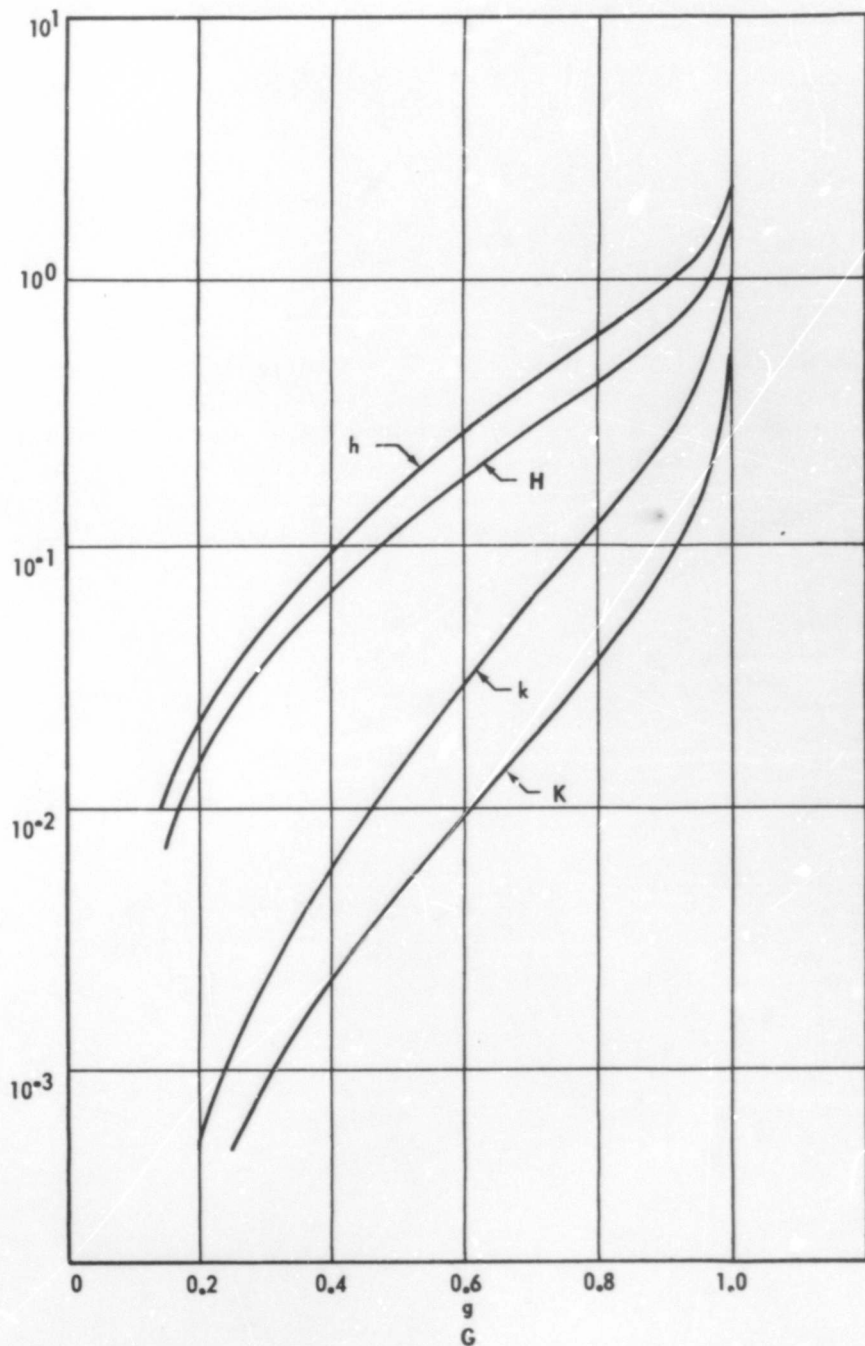
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FIGURE 1. WEIGHT RELATIONSHIPS OF OPTIMUM (LOWER CASE)
AND CRITICALLY DAMPED (UPPER CASE) FILTERS

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_M^2 \end{bmatrix}, \qquad (10)$$

may be defined.

Element $Q_{33}$ of this matrix $\left(\sigma_M^2\right)$ is actually the maneuver variance. It represents the uncertainty in predicted acceleration caused by derivatives in the trajectory higher than the second, during one sample interval. Element $Q_{33}$ may be approximated from the maximum target jerk by neglecting derivatives higher than the third. If the maximum jerk of a target is 5 g per second, this, multiplied by T, might be taken to be the $3\sigma$ maneuver and when broken down into its coordinate components, one of which is the filter input, $\sigma_M^2$ is obtained.

Also a sampled data system transition matrix for a constant acceleration input must be defined. It is:

$$\Phi = \begin{bmatrix} 1 & T & \dfrac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}. \qquad (11)$$

From Schwarz and Friedland [1] [Section 4.2, page 107, Equation (4.5)] $\Phi(t)$ is defined as:

$$\Phi(t) = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots \quad .$$

However since

$$\ddot{y} = \text{constant}$$

then

$$\dddot{y} = 0 .$$

Therefore, Equation (4.3) in the reference becomes:

$$\dot{y}(t) = Ay(t) ,$$

or

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \\ \dddot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} .$$

Thus

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

And from Equation (4.5), with $I$ = the identity matrix,

$$\Phi(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} T + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{T^2}{2} .$$

Therefore

$$\Phi(T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & T & 0 \\ 0 & 0 & T \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{T^2}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} .$$

Now, two other matrices are defined which will be used later;

$$W = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} \tag{12}$$

is the second order filter "weighting" matrix;

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{13}$$

is termed the observation matrix and relates the state vector $[Y]$ to the measurement vector $[X]$. The number of rows in the observation matrix indicates the number of ways one intends to measure independently a state variable such as position. A non-zero element indicates what variable or

variables are being measured. The number of columns corresponds to the number of rows in the state vector. Therefore, it can be seen that since the only input is position $[X_1]$, being measured in only one way, the observation matrix is defined as in Equation (13).

There are two other matrices which must be defined. These are the smoothed error covariance matrix and the predicted error covariance matrix.

$$\hat{P}(n+1) = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} & \hat{P}_{13} \\ \hat{P}_{21} & \hat{P}_{22} & \hat{P}_{23} \\ \hat{P}_{31} & \hat{P}_{32} & \hat{P}_{33} \end{bmatrix},$$

where $\hat{P}_{11}$ is the predicted position covariance, or

$$\hat{P}_{11} = E[(\hat{y} - E\hat{y})(\hat{y} - E\hat{y})] = E[(\hat{y} - E\hat{y})^2] ,$$

where E denotes the expected value. Therefore, each of the elements of $\hat{P}_{(n+1)}$ is defined (but not calculated) as follows:

$$\hat{P}_{12} = E[(\hat{y} - E\hat{y})(\dot{\hat{y}} - E\dot{\hat{y}})]$$
$$\hat{P}_{13} = E[(\hat{y} - E\hat{y})(\ddot{\hat{y}} - E\ddot{\hat{y}})]$$
$$\hat{P}_{23} = E[(\dot{\hat{y}} - E\dot{\hat{y}})(\ddot{\hat{y}} - E\ddot{\hat{y}})]$$
$$\hat{P}_{22} = E[(\dot{\hat{y}} - E\dot{\hat{y}})(\dot{\hat{y}} - E\dot{\hat{y}})] = E[(\dot{\hat{y}} - E\dot{\hat{y}})^2]$$
$$\hat{P}_{33} = E[(\ddot{\hat{y}} - E\ddot{\hat{y}})(\ddot{\hat{y}} - E\ddot{\hat{y}})] = E[(\ddot{\hat{y}} - E\ddot{\hat{y}})^2] .$$

The same is true for $\bar{P}_{(n+1)}$ excepting the fact that ($\char`\^$) is replaced by ($-$). Note at this point that the matrices $\bar{P}$ and $\hat{P}$ are symmetrical about the main diagonal. This will be a major simplifying factor brought out later.

Now the Kalman sequence and the general Kalman equations for a second-order filter may be introduced. Once again the symbol ($\char`\^$) means predicted and ($-$) means smoothed quantities. The general equation for determining weights is

$$W(n+1) = \hat{P}(n+1) H^T \left[ H \hat{P}(n+1) H^T + R \right]^{-1} . \tag{14}$$

All matrices except the measurement variance have been defined:

$$R = \left[\sigma_R{}^2\right] , \qquad (15)$$

where $\sigma_R$ is the standard deviation of the radar errors. Thus R, in this case, is a $1 \times 1$ matrix.

The other two equations needed to update the covariances and calculate the weights are:

$$\overline{P}(n+1) = [I - WH] \; \hat{P}(n+1) \qquad (16)$$

$$\hat{P}(n+1) = \Phi \; \overline{P}(n) \; \Phi^T + Q . \qquad (17)$$

Equations (16) and (17) will be combined to form the error covariance matrix for the predicted state:

$$\hat{P}(n+1) = \Phi \; [I - WH] \; \hat{P}(n) \; \Phi^T + Q . \qquad (18)$$

The above equation may be written in expanded matrix form and the matrix algebra carried out, since all the matrices have been defined:

$$\hat{P}(n+1) = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \left\{ \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} [1\ 0\ 0] \right] \begin{bmatrix} \hat{P}_{11}(n) & \hat{P}_{12}(n) & \hat{P}_{13}(n) \\ \hat{P}_{21}(n) & \hat{P}_{22}(n) & \hat{P}_{23}(n) \\ \hat{P}_{31}(n) & \hat{P}_{32}(n) & \hat{P}_{33}(n) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ \frac{T^2}{2} & T & 1 \end{bmatrix} \right\} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_M{}^2 \end{bmatrix} .$$

$$(19)$$

At this point it becomes advantageous to get explicit expressions for the weights. From Equation (14):

$$W(n+1) = \begin{bmatrix} \hat{P}_{11}(n+1) & \hat{P}_{12}(n+1) & \hat{P}_{13}(n+1) \\ \hat{P}_{21}(n+1) & \hat{P}_{22}(n+1) & \hat{P}_{23}(n+1) \\ \hat{P}_{31}(n+1) & \hat{P}_{32}(n+1) & \hat{P}_{33}(n+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \left\{ [1\ 0\ 0] \begin{bmatrix} \hat{P}_{11}(n+1) & \hat{P}_{12}(n+1) & \hat{P}_{13}(n+1) \\ \hat{P}_{21}(n+1) & \hat{P}_{22}(n+1) & \hat{P}_{23}(n+1) \\ \hat{P}_{31}(n+1) & \hat{P}_{32}(n+1) & \hat{P}_{33}(n+1) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + [R] \right\}^{-1} .$$

Therefore,

$$W(n+1) = \begin{bmatrix} W_1(n+1) \\ W_2(n+1) \\ W_3(n+1) \end{bmatrix} = \begin{bmatrix} \hat{P}_{11}(n+1)/[\hat{P}_{11}(n+1) + R] \\ \hat{P}_{12}(n+1)/[\hat{P}_{11}(n+1) + R] \\ \hat{P}_{13}(n+1)/[\hat{P}_{11}(n+1) + R] \end{bmatrix} \quad . \tag{20}$$

It is now apparent that:

$$W_1(n+1) = \hat{P}_{11}(n+1)/[\hat{P}_{11}(n+1) + R] \tag{21}$$

$$W_2(n+1) = \hat{P}_{12}(n+1)/[\hat{P}_{11}(n+1) + R] \tag{22}$$

$$W_3(n+1) = \hat{P}_{13}(n+1)/[\hat{P}_{11}(n+1) + R] \quad . \tag{23}$$

These three equations may be used quite extensively in simplifying the equations which result from expanding Equation (19). It may now be shown from Equation (19) that:

$$\hat{P}_{11}(n+1) = W_1 R + 2 W_2 RT + W_3 RT^2 - \hat{P}_{12}(n) W_2 T^2 - W_2 \hat{P}_{13}(n) T^3$$
$$+ \hat{P}_{22}(n) T^2 + \hat{P}_{23}(n) T^3 - \frac{1}{4} \hat{P}_{13}(n) W_3 T^4 + \frac{1}{4} \hat{P}_{33}(n) T^4 \tag{24}$$

$$\hat{P}_{12}(n+1) = W_2 R + W_3 RT - \hat{P}_{12}(n) W_2 T + \hat{P}_{22}(n) T + \frac{3}{2} \hat{P}_{23}(n) T^2$$
$$- \frac{3}{2} \hat{P}_{12}(n) WT^2 - \frac{1}{2} \hat{P}_{13}(n) W_3 T^3 + \frac{1}{2} \hat{P}_{33}(n) T^3 \tag{25}$$

$$\hat{P}_{13}(n+1) = W_3 R - \hat{P}_{12}(n) W_3 T + \hat{P}_{23}(n) T + \frac{1}{2} \hat{P}_{33}(n) T^2 - \frac{1}{2} \hat{P}_{13}(n) W_3 T^2 \tag{26}$$

$$\hat{P}_{22}(n+1) = - \hat{P}_{12}(n) W_2 + \hat{P}_{22}(n) + 2 \hat{P}_{23}(n) T - 2 \hat{P}_{12}(n) W_3 T$$
$$- \hat{P}_{13}(n) W_3 T^2 + \hat{P}_{33} T^2 \tag{27}$$

$$\hat{P}_{33}(n+1) = - \hat{P}_{13}(n) W_3 + \hat{P}_{33}(n) + \sigma_M^2 \tag{28}$$

$$\hat{P}_{23}(n+1) = - \hat{P}_{12}(n) W_3 + \hat{P}_{23}(n) - \hat{P}_{13}(n) W_3 T + \hat{P}_{33}(n) T \quad . \tag{29}$$

In the steady state the covariance and weights approach constants. Since this is true,

8

$$\hat{P}_{ij}(n+1) = \hat{P}_{ij}(n) \ ,$$

and the time indexes may be dropped. The resulting equations fix the optimum steady-state weights.

## 3. Relationship Between Optimum and Critically Damped Filter

The previous six equations (24-29) may now be manipulated to obtain relationships between the three weights ($W_1$, $W_2$, and $W_3$) by eliminating the P's which cannot be described in terms of the weights and substituting for those P's which can. Equation (24) is multiplied by two, Equation (25) by minus two T (-2T) and Equation (26) by T squared ($T^2$), and the resulting equations are combined. From Equations (21), (22), and (23) it can readily be seen that:

$$2TW_2 = W_1\left(W_1 + W_2 T + W_3 \frac{T^2}{2}\right) \tag{30}$$

Considering Equations (26) and (27) in a similar manner, one may deduce the equation:

$$W_2^{\ 2} = 2W_1 W_3 \ . \tag{31}$$

Now Equation (28) easily reduces to:

$$\frac{W_3^{\ 2}}{1-W_1} = \frac{\sigma_M^{\ 2}}{R} \ . \tag{32}$$

Equations (30), (31), and (32) are the basic weight relationships of the optimum steady-state filter.

Now the second-order Kalman needs to be related to the previously described G-H-K. The Kalman smoothing and prediction equations (with no control vector) are:

$$[\widehat{Y(n+1)}] = [\Phi]\ [\overline{Y}(n)\,]$$

$$[\overline{Y}(n)\,] = [\widehat{Y}(n)\,] + [W]\left\{[X(n)\,] - [H][\widehat{Y}(n)\,]\right\}$$

or

$$\hat{y}(n+1) = \overline{y}(n) + \overline{\dot{y}}(n)T + \overline{\ddot{y}}(n)\frac{T^2}{2}$$

$$\hat{\dot{y}}(n+1) = \overline{\dot{y}}(n) + \overline{\ddot{y}}(n)T$$

$$\hat{\ddot{y}}(n+1) = \overline{\ddot{y}}(n)$$

9

$$\bar{y}(n) = \hat{y}(n) + W_1\left[x(n) - \hat{y}(n)\right]$$

$$\bar{\dot{y}}(n) = \hat{\dot{y}}(n) + W_2\left[x(n) - \hat{y}(n)\right]$$

$$\bar{\ddot{y}}(n) = \hat{\ddot{y}}(n) + W_3\left[x(n) - \hat{y}(n)\right] \quad .$$

Thus it becomes apparent that the following relationships hold true for optimum steady-state second order, and critically damped second-order filters:

$$W_1 = g \tag{33}$$

$$W_2 T = h \tag{34}$$

$$W_3 \frac{T^2}{2} = k \quad . \tag{35}$$

Weights (optimum) h and k are shown as functions of g in lower case letters in Figure 1. Shown in Figure 2 is a curve relating the optimum weight g to the parameter $T^4 \dfrac{\sigma_M^2}{\sigma_R^2}$. Thus, the weights for the optimum filter are fixed if T, $\sigma_M$, and $\sigma_R$ are known.

In summary, there is actually no difference in the critically damped and the optimum filters except in the calculation of the weights.

## 4. Variance Reduction Ratio

Attention is now turned to the criteria to be used in comparing the two second-order filters. It can be shown that there exists a variance reduction ratio, $\delta^2$, which, for predicted position, is a function only of the weights of the second-order filter. This function is defined as follows:

$$\delta^2 = \frac{\sigma_{\hat{y}}^2}{\sigma_x^2} = \frac{gk(2g + h - 4) + h\left[g\,(2g + h) + 2h\right]}{\left[-g(h + k) + 2k\right](2g + h - 4)} \quad ,$$

where $\sigma_{\hat{y}}^2$ and $\sigma_x^2$ are the variances of the output and the input respectively. The actual step-by-step derivation of this term is beyond the scope of this paper, however, the procedure[1] will be pointed out. Using Equations (1) through (6), one may solve for every element of the covariance matrix with the weights

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[1] C. F. Asquith, unpublished notes.
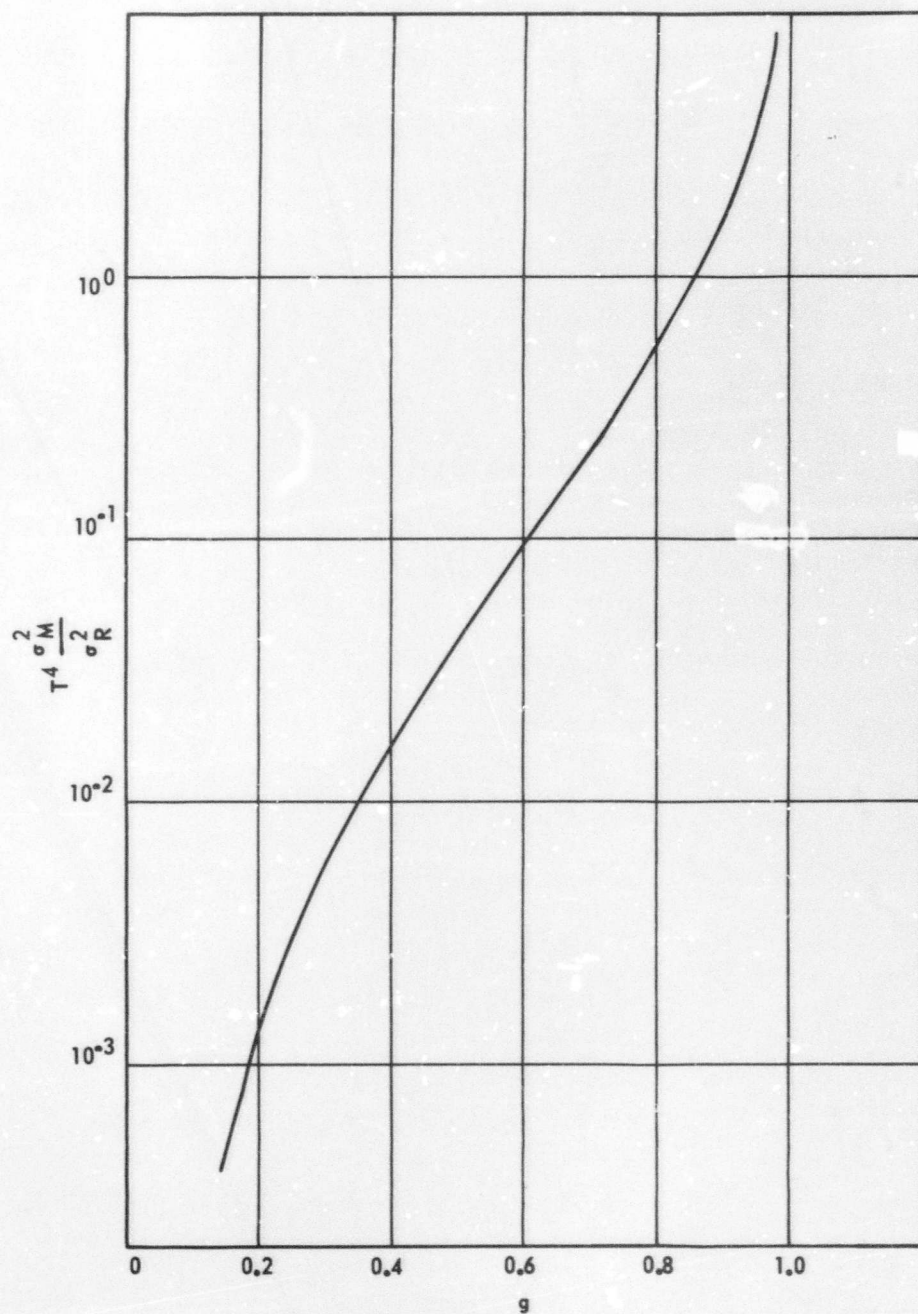
FIGURE 2. KALMAN STEADY-STATE "OPTIMUM"

constant by calculating $E[(y - Ey)^2]$, $E[(y - Ey)(\dot{y} - E\dot{y})]$, etc. Then the indexes may be dropped, making the steady-state assumption. Next, $P_{12}$, $P_{13}$, $P_{22}$, $P_{23}$, and $P_{33}$ must all be eliminated, leaving an equation for $P_{11}/R$ which is actually $\dfrac{\sigma_{\hat{y}}^2}{\sigma_x^2}$ for predicted position only. (The same procedure may be followed if the variance reduction ratio is desired for either velocity or acceleration.)

Therefore, by simply knowing the weights, one may calculate the noise ratio of the two filters. The noise ratio, $\delta^2$, for the two filters may then be equalized and the filters may be compared on other grounds.

### 5. Truncation Error

The criterion that has been chosen for comparison of the two filters is an error which arises from the inputting to the two filters, of a curve of one degree higher than the order of the filter. In the case of the second-order filter, this would be an input comprised of a change in acceleration $(\dddot{x})$ or "jerk." When this is input into a second-order filter, the steady-state error which arises is termed truncation error.

Assuming a trajectory with constant jerk $(J_0)$, the acceleration, velocity, and position can be calculated as functions of time. The acceleration equals the total time times the jerk or:

$$\text{Acceleration} = nT J_0 . \tag{36}$$

Integrating this expression, the velocity and position are obtained:

$$\text{Velocity} = \frac{nT^2}{2} J_0 \tag{37}$$

$$\text{Position} = \frac{nT^3}{6} J_0 . \tag{38}$$

The noise-free measured values at times n, n-1, and n-2 are denoted as:

$$x_n = \frac{nT^3}{6} J_0 \tag{39}$$

$$x_{n-1} = \frac{(n-1)^3 T^3}{6} J_0 \tag{40}$$

$$x_{n-2} = \frac{(n-2)^3 T^3}{6} J_0 . \tag{41}$$

12

Now the equation for linear feedback filters, in this case, Equation (7), is recalled:

$$y(n+1) = (g + h + k) \; x_n + (-2g - h + k) \; x_{n-1}$$
$$+ g \; x_{n-2} - (-3 + g + h + g) \; y_n$$
$$- (3 - 2g - h + k) \; y_{n-1} - (-1 + g) \; y_{n-2} \qquad (7)$$

The errors then, in the predicted, are defined for the present point and the past three as:

$$\epsilon(n+1) = y(n+1) - \frac{(n+1)^3 \; T^3 \; J_0}{6} \qquad (42)$$

$$\epsilon(n) = y(n) - \frac{n^3 \; T^3 \; J_0}{6} \qquad (43)$$

$$\epsilon(n-1) = y(n-1) - \frac{(n-1)^3 \; T^3 \; J_0}{6} \qquad (44)$$

$$\epsilon(n-2) = y(n-2) - \frac{(n-2)^3 \; T^3 \; J_0}{6} \quad . \qquad (45)$$

Solving Equations (42) through (45) for the respective y's and substituting them in Equation (7), one obtains:

$$\epsilon(n+1) + (n+1)^3 \; T^3 \frac{J_0}{6} = (g + h + k) \; (nT)^3 \frac{J_0}{6} + (-2g - h + k) \; (n-1)^3 \; T^3 \frac{J_0}{6}$$

$$+ g \; (n-2)^3 \; T^3 \frac{J_0}{6} - (-3 + g + h + k) \left( \epsilon(n) + n^3 \; T^3 \frac{J_0}{6} \right) - (3 - 2g - h + k)$$

$$\left[ \epsilon(n-1) + (n-1)^3 \; T^3 \frac{J_0}{6} \right] - (-1 + g) \left[ \epsilon(n-2) + (n-2)^3 \; T^3 \frac{J_0}{6} \right] \quad .$$

Simplifying, the previous equation becomes:

$$2k \; \epsilon(n+1) = T^3 \frac{J_0}{6} \; [ -(n+1)^3 + (g + h + k) \; n^3 + (-2g - h + k) \; (n-1)^3$$

$$+ g \; (n-2)^3 - (-3 + g + h + k) \; n^3 - (3 - 2g - h + k) \; (n-1)^3 - (-1 + g) \; (n-2)^3 ] \quad .$$

13

Therefore:

$$2k \; \epsilon(n+1) = T^3 \frac{J_0}{6} \; [-(n+1)^3 + 3 \; n^3 - 3(n-1)^3 + (n-2)^3] \quad .$$

Finally the equation reduces to:

$$2k \; \epsilon(n+1) - T^3 \frac{J_0}{6} \; [6] = - T^3 J_0 \quad .$$

Thus:

$$\epsilon(n+1) = \frac{- T^3 J_0}{2 \; k} \quad .$$

In summary, the truncation error is a function of $\frac{1}{k}$. Therefore, in order to minimize truncation error, k must be maximized.

Now the filter weights must be chosen so that the noise ratios of the two filters correspond. Then the truncation errors of the two filters can be compared to determine which, the critically damped or the optimum, has less truncation error from the constant jerk input (Figure 3).

## 6. Conclusions

It is readily apparent that the optimum filter offers a far better minimization of truncation error than the critically damped filter. In fact, choosing a certain noise reduction ratio one may determine from Figure 3 that there is approximately a 40-percent decrease in truncation error.

It must be noted that the truncation error due to those inputs of two or more degrees higher than that of the second-order filter have been neglected. These derivatives generally are of little significance, however, in the calculation of total truncation error because these inputs would be very small and of short duration.

Therefore, in closing, if the problem is to minimize truncation error, there is a marked advantage in using the optimum, second-order filter over the critically damped second-order filter. Applicable data and discussion have been given in previous works [2-4].
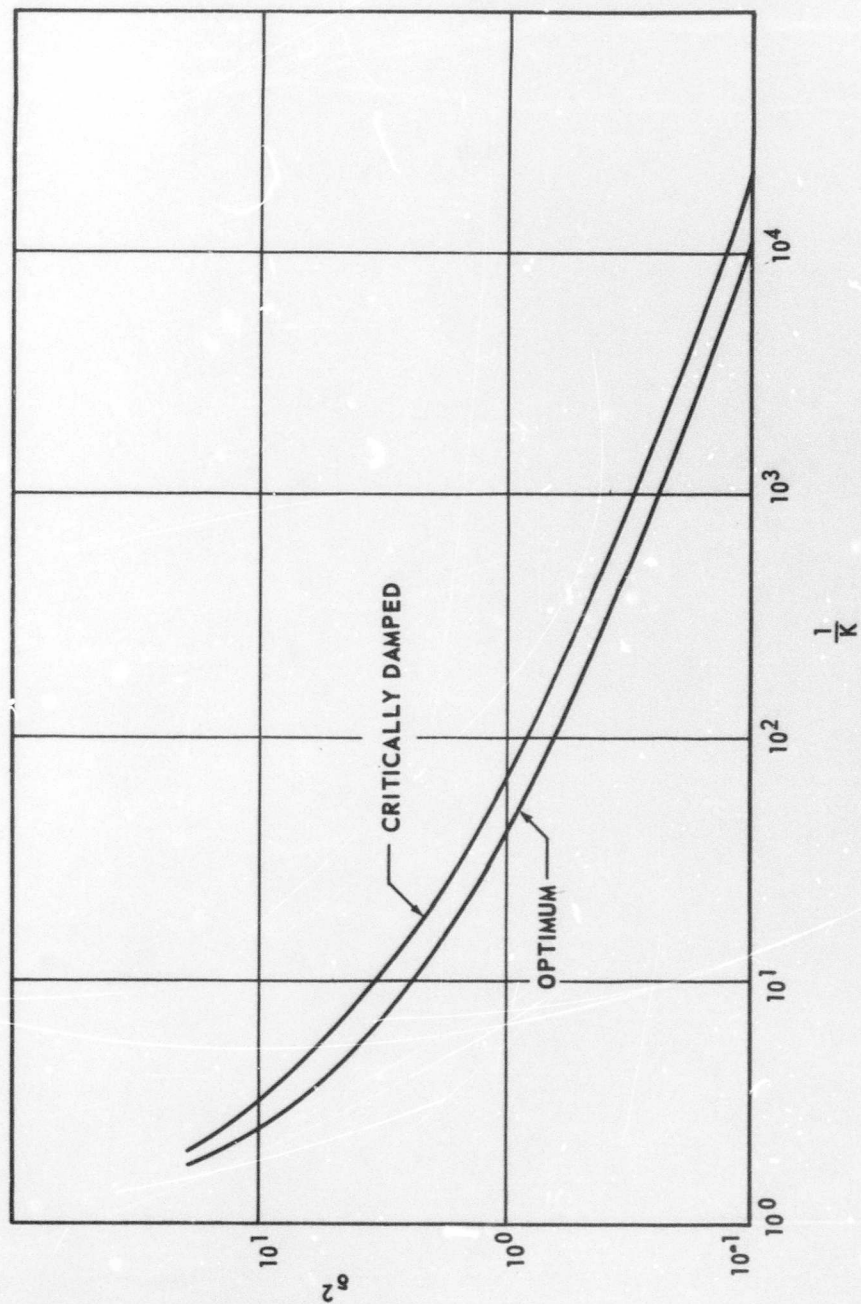
14

FIGURE 3. CRITICALLY DAMPED AND OPTIMUM FILTERS: TRUNCATION ERRORS

15

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4.  J. R. Ragazzini and G. F. Franklin, <u>Sampled Data Control Systems</u>, McGraw-Hill, New York, 1958.

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13. ABSTRACT

    Second-order, linear, recursive, fixed-weight optimal and critically damped
filters are compared on the basis of truncation errors produced by each. The critically
damped and optimal filters are derived along with the commonly used parameters which
are truncation error and noise ratio. It is shown that when truncation errors are
compared on the basis of equal noise ratios for the two filters, the optimal filter yields
a significant improvement over the critically damped filter.

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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
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| | ROLE | WT | ROLE | WT | ROLE | WT |
| Fixed-weight optimal filters<br>Critically damped filters<br>Truncation errors<br>Noise ratio | | | | | | |